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Estimates of the Hermite and the Freud Polynomials

STANFORD S. BONAN AND DEAN S. CLARK

Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881, U.S.A.

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INTRODUCTION

The Hermite polynomials $\{p_n(x)\}_{n=0}^{\infty}$ may be defined by their orthogonality property as

$$\int_{-\infty}^{\infty} p_n(x)\, p_k(x) \exp(-x^2)\, dx = \begin{cases} 0, & n \neq k, \\ 1, & n = k, \end{cases}$$

and

$$p_n(x) = \gamma_n x^n + \cdots, \qquad \text{where} \quad \gamma_n > 0.$$

These polynomials as well as other classical orthogonal polynomials have been widely studied. In this paper a new way of obtaining pointwise estimates of $p_n(x)$ is explained. It is then shown that the method applies for a more general class of orthogonal polynomials, the so-called Freud polynomials.

We list exactly the type of estimates we are aiming for in

THEOREM 1. *Let $p_n(x)$ denote the Hermite polynomial of degree n. Then there exist positive constants C, D, and E such that*

(i) $p_n^2(x) \exp(-x^2) \leqslant C/\sqrt{2n+1-x^2}$ *when* $|x| \leqslant \sqrt{2n+1}$,

(ii) $\max_{x \in \mathbb{R}} p_n^2(x) \exp(-x^2) \leqslant Dn^{-1/6}$, *and*

(iii) $\max_{x \in \mathbb{R}} p_n^2(x) \exp(-x^2) \geqslant En^{-1/6}$, *for* $n = 1, 2, 3, \ldots$

These results are known from more informative asymptotic formulas (see Erdélyi [2]). Our approach to these estimates will be stated below and generalized to the Freud polynomials later.

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THE STRATEGY

It is well known that the Hermite polynomials satisfy a differential equation of the form

$$z''(x) + \phi(x, n)\, z(x) = 0,$$

where

$$z(x) = p_n(x) \exp(-x^2/2),$$

and

$$\phi(x, n) = 2n + 1 - x^2.$$

The key observation in verifying our estimates of the Hermite polynomials is this: Whenever the product $\phi(x, n)\, z(x)$ is positive, the graph of $y = z(x)$ is concave down and we have the inequality

$$z(t) \cdot \frac{(b-a)}{2} \leqslant \int_a^b z(x)\, dx, \qquad a < t < b,$$

when $z(x)$ is positive. The inequality simply says that the area under the curve z from a to b is larger than the area of the triangle with height $z(t)$, whose base on the x-axis has length $b - a$.

To estimate the above integral, where $a < b$ are consecutive zeros of p_n, we make a three-way attack. First we use Schwartz's inequality, then Sturm's estimate for consecutive zeros of $z(x)$ in estimating the quantity $b - a$. Finally, basic identities from the othogonality of the Hermite polynomials allow estimates of the resulting integrals and produce part (i) of the theorem. To complete the results of the theorem we find that (ii) follows immediately from the estimates produced in (i). But part (iii) presents technical complications because our approach involves estimating the kernel function

$$K_n(x) = \sum_{j=0}^{n-1} p_j^2(x).$$

THE PROOF

From the differential equation we observe the following:

COROLLARY 1. *Let* $|x| \leqslant \sqrt{2n+1}$. *Then for some zero* x_{kn} *of* $p_n(x)$ *we have*

$$|x - x_{kn}| \leqslant \frac{\pi}{\sqrt{2n + 1 - x^2}}.$$

Proof. Since we will use Sturm's Comparison Theorem, we introduce the differential equation

$$z_t''(x) + (2n+1-t^2)\, z_t(x) = 0,$$

so that

$$2n+1-t^2 \leqslant 2n+1-x^2 \qquad \text{when} \quad |x| \leqslant t \leqslant \sqrt{2n+1}.$$

Sturm's Theorem says that the zeros of $z(x)$ separate the zeros of $z_t(x)$. But since $z_t(x) = \sin(\sqrt{2n+1-t^2})x$ is a solution to the above differential equation, which has zeros

$$x_k = \frac{k\pi}{\sqrt{2n+1-t^2}}, \qquad k = 0, \pm 1, \pm 2, ...,$$

we get

$$|x - x_{kn}| \leqslant \frac{\pi}{\sqrt{2n+1-t^2}}$$

for some zero x_{kn} of $p_n(x)$. The corollary follows by noting that t is arbitrary except for the inequality $|x| \leqslant |t| \leqslant \sqrt{2n+1}$. ∎

Another important relationship is the closeness of x_{1n}, the largest zero of $p_n(x)$ to $\sqrt{2n+1}$.

COROLLARY 2. $|\sqrt{2n+1} - x_{1n}| \leqslant (\pi+1)n^{-1/6}$, *for* $n = 1, 2, ...$

Proof. From the differential equation we have $x_{1n} < \sqrt{2n+1}$. Letting $x = \sqrt{2n+1} - n^{-1/6}$, one uses Corollary 1 to complete the proof. ∎

Note that the number $\pi + 1$ appearing in Corollary 2 is not important for us: our main concern is the factor $n^{-1/6}$ which appears.

The reader will recall our strategy calls for some basic inequalities from orthogonal polynomials. The first identity is the Gauss–Jacobi formula (for a proof see Szegö [8]).

THEOREM 2. *For any polynomial* π_{2n-1} *of degree at most* $2n-1$

$$\int_{-\infty}^{\infty} \pi_{2n-1}(x) \exp(-x^2)\, dx = \sum_{k=1}^{n} \pi_{2n-1}(x_{kn})\, \lambda_{kn}$$

for some numbers λ_{kn}, $k = 1, 2, ..., n$, *which do not depend on the particular polynomial* π_{2n-1}.

Using the Gauss–Jacobi formula and the identity (see Szegö [8])

$$p_n'(x) = \sqrt{2n}\, p_{n-1}(x)$$

relating the Hermite polynomial and its derivative gives us the following corollary whose proof is due to G. Freud (see [3]).

COROLLARY 3. *Let $p_n(x)$ denote the Hermite polynomial of degree n, then*

$$2 = p_n'^2(x_{kn})\, \lambda_{kn}.$$

Proof. Let

$$\pi_{2n-1}(x) = \frac{p_n(x)\, p_{n-1}(x)}{x - x_{kn}}$$

in Theorem 2. A direct evaluation of each side of the equation gives

$$\frac{\gamma_{n-1}}{\gamma_n} = p_n'(x_{kn})\, p_{n-1}(x_{kn})\, \lambda_{kn}.$$

Corollary 3 now follows from the identity relating the Hermite polynomial and its derivative. ∎

We are now prepared to prove part (i) of our theorem.

Proof (Part (i)*).* Let $|x| \leqslant \sqrt{2n+1}$. By the concavity of $z(x)$ we can compare areas as explained earlier and get

$$\frac{1}{2} z(x)(x - x_{kn}) \leqslant \int_{x_{kn}}^{x} z(t)\, dt.$$

By Schwartz's inequality

$$\left\{\frac{1}{2} z(x)(x - x_{kn})\right\}^2 \leqslant \left(\int_{-\infty}^{\infty} \frac{p_n^2(x) \exp(-x^2)}{(x - x_{kn})^2}\, dx\right)\left(\int_{x_{kn}}^{x} (t - x_{kn})^2\, dt\right)$$
$$= p_n'^2(x_{kn})\, \lambda_{kn} \frac{(x - x_{kn})^3}{3}.$$

Therefore, from the above and Corollary 3,

$$z^2(x) \leqslant \tfrac{8}{3}(x - x_{kn}).$$

The proof of part (i) now follows from Corollary 1. ∎

For the proof of part (ii) of our theorem, observe that, since $z(x)$ decreases when $x > \sqrt{2n+1}$, we may use part (i) when $|x| \leqslant x_{1n}$, then use the last inequality in the proof of part (i) when $x_{1n} \leqslant |x| \leqslant \sqrt{2n+1}$. Part (ii) now follows from Corollary 2, the estimate of x_{1n}.

Our technique for verifying part (iii) of our theorem is based on the identity

$$1 = \int_{-\infty}^{\infty} \left(\frac{p_n(x)}{x - x_{1n}}\right)^2 \frac{\exp(-x^2)}{p_n'^2(x_{1n})\,\lambda_{1n}}\,dx,$$

which is a direct result of the Gauss–Jacobi formula. We will show that a significant amount of the integral occurs away from x_{1n}. That is, for some $\varepsilon > 0$, we claim that

$$\frac{1}{2} \leqslant \int_{|x - x_{1n}| \geqslant \varepsilon n^{-1/6}} \left(\frac{p_n(x)}{x - x_{1n}}\right)^2 \frac{\exp(-x^2)}{p_n'^2(x_{1n})\,\lambda_{1n}}\,dx, \qquad (*)$$

which implies

$$\tfrac{1}{2} \leqslant \max_{x \in \mathbb{R}} p_n^2(x) \exp(-x^2)(\varepsilon n^{-1/6})^{-1},$$

completing part (iii) of our theorem.

In order to show that (*) holds for some $\varepsilon > 0$ we first mention a basic inequality from orthogonal polynomials (see Szegö [8]).

THEOREM 3.

$$\pi_{n-1}^2(x) \leqslant \left(\int_{-\infty}^{\infty} \pi_{n-1}^2(t) \exp(-t^2)\,dt\right)\left(\sum_{k=0}^{n-1} p_k^2(x)\right),$$

for any polynomial π_{n-1} *of degree at most* $n-1$.

We can now estimate that part of the integral near x_{1n} by

$$\begin{aligned}
&\int_{|x - x_{1n}| \leqslant \varepsilon n^{-1/6}} \left(\frac{p_n(x)}{x - x_{1n}}\right)^2 \frac{\exp(-x^2)}{p_n'^2(x_{1n})\,\lambda_{1n}}\,dx \\
&\quad \leqslant \int_{|x - x_{1n}| \leqslant \varepsilon n^{-1/6}} \left(\sum_{k=0}^{n-1} p_k^2(x)\right) \exp(-x^2)\,dx \\
&\quad \leqslant 2\varepsilon n^{-1/6} \max_{|x - x_{1n}| \leqslant \varepsilon n^{-1/6}} \left(\sum_{k=0}^{n-1} p_k^2(x)\right) \exp(-x^2).
\end{aligned}$$

This leads us to the problem of estimating the Kernel function

$$K_n(x) = \sum_{k=0}^{n-1} p_k^2(x)$$

in the interval $|x - x_{1n}| \leqslant \varepsilon n^{-1/6}$. We outline this task as follows. If $|x| \leqslant \sqrt{2k+1}$ we may use our estimates in part (i) to estimate $p_k^2(x)$, while if x is near $\sqrt{2k+1}$ we use the result of part (ii). But if $|x| \geqslant \sqrt{2k+1}$ we will need another way of estimating $p_k^2(x)$. The way we will accomplish this will be the following.

Consider

$$z'(x) = \int_{x_0}^{x} z''(t)\, dt,$$

where $z'(x_0) = 0$ and $x_0 \in [x_{1n}, \sqrt{2n+1}]$. Since $z(x)$ decreases for $x > \sqrt{2n+1}$,

$$\int_{x_0}^{x} (t^2 - (2n+1))\, z(t)\, dt < 0 \qquad \text{for} \quad x > \sqrt{2n+1}.$$

Examining where the integral is positive and negative leads to

$$\int_{x_0}^{\sqrt{2n+1}} (2n+1-t^2)\, z(t)\, dt > \int_{\sqrt{2n+1}}^{x} (t^2 - (2n+1))\, z(t)\, dt.$$

To complete the pointwise estimate of $z(x)$ for $|x| > \sqrt{2n+1}$ we use part (ii) of our theorem to bound $\max_{t \in \mathbb{R}} z(t)$, and the estimate of x_{1n} in Corollary 2 to estimate x_0 in the left hand side integral above. This gives

LEMMA 1. *For some positive constant F*

$$|z(x)| \leqslant \frac{Fn^{-5/12}}{(x - \sqrt{2n+1})^2} \qquad \textit{when} \quad |x| > \sqrt{2n+1}.$$

Now we bound the kernel function $K_n(x)$ using Lemma 1 and parts (i) and (ii) of our theorem, and get

$$\max_{|x - x_{1n}| \leqslant \varepsilon n^{-1/6}} \left(\sum_{k=0}^{n-1} p_k^2(x) \right) \exp(-x^2) \leqslant G n^{1/6},$$

for some positive constant G. Thus we see that (*) holds when $\varepsilon < 1/2G$, and part (iii) is verified.

GENERALIZING TO THE FREUD POLYNOMIALS

In order to define the Freud polynomials we simply go back and change the x^2 to x^m with m an even positive integer:

$$\int_{-\infty}^{\infty} p_n(x)\, p_k(x) \exp(-x^m)\, dx = \begin{cases} 0, & n \neq k, \\ 1, & n = k, \end{cases}$$

where $p_n(x)$ has positive leading coefficient γ_n.

Geza Freud investigated these polynomials and their properties extensively, and recently others have studied them. Some studies and surveys may be found in [1, 4, 6, 7]. Henceforth we use $p_n(x)$ for this more general case.

We will presently see that our method applied to the Hermite polynomials also works for the Freud polynomials. However, some of the details are more complicated, and we need to consider some formulas in more detail, formulas we have learned to take for granted in the Hermite case.

For a statement of the results we refer the reader to the end of the paper, although it is a good idea to note where the "Freud constant" is defined, below.

Differential Properties

It is convenient to start with the recurrence formula

$$xp_{n-1}(x)=\frac{\gamma_{n-1}}{\gamma_n}\,p_n(x)+\frac{\gamma_{n-2}}{\gamma_{n-1}}\,p_{n-2}(x).$$

Now what is γ_{n-1}/γ_n? Freud conjectures a value of γ_{n-1}/γ_n, and finally Máté, Nevai, and Zaslavsky [5] established an asymptotic series. We will use their first term and a convenient error term in estimating γ_{n-1}/γ_n.

Theorem 4.

$$\frac{\gamma_{n-1}}{\gamma_n}=\beta n^{1/m}+\varepsilon_n,$$

where

$$|\varepsilon_n|\leqslant Hn^{-1+1/m}\qquad \textit{for}\quad n=1,2,...$$

and some positive constant H. Here β denotes Freud's constant

$$\beta=\tfrac{1}{2}\{\pi^{1/2}\Gamma(m/2)/\Gamma(m+1/2)\}^{1/m}.$$

To facilitate the presentation we introduce a new notation to denote the relationship in the first two lines of Theorem 4. The expression

$$\frac{\gamma_{n-1}}{\gamma_n}=[\beta n^{1/m}]$$

abbreviates the statement in Theorem 4. More specifically,

$$a_n=[f(n)]$$

means that for some sequence $\{\varepsilon_n\}_{n=1}^{\infty}$

$$a_n = f(n) + \varepsilon_n,$$

where

$$|\varepsilon_n| \leqslant Hn^{-1+1/m} \qquad \text{for} \quad n = 1, 2, \ldots \text{ and } H > 0.$$

With the above notation the recurrence formula takes the form

COROLLARY 4. $xp_{n-1}(x) = [\beta n^{1/m}]\, p_n(x) + [\beta n^{1/m}]\, p_{n-2}(x)$.

A simple consequence of Corollary 4 is

COROLLARY 5.

$$x^r p_n(x) = \sum_{j=0}^{r} \binom{r}{j} [\beta n^{1/m}]^r\, p_{n+2j-r}(x),$$

for any nonnegative integer r.

Proof. If $r = 0$ we get $p_n(x) = p_n(x)$. Proceeding inductively, consider $r > 0$

$$\begin{aligned}
x^r p_n(x) &= x(x^{r-1} p_n(x)) \\
&= x \sum_{j=0}^{r-1} \binom{r-1}{j} [\beta n^{1/m}]^{r-1} p_{n+2j-(r-1)}(x) \\
&= \sum_{j=0}^{r-1} \binom{r-1}{j} [\beta n^{1/m}]^{r-1} ([\beta n^{1/m}]\, p_{n+2j-r+2}(x) \\
&\quad + [\beta n^{1/m}]\, p_{n+2j-r}(x)) \\
&= \sum_{j=0}^{r-1} \binom{r-1}{j} [\beta n^{1/m}]^r\, p_{n+2j-r} \\
&\quad + \sum_{j=1}^{r} \binom{r-1}{j-1} [\beta n^{1/m}]^r\, p_{n+2j-r}(x).
\end{aligned}$$

The verification is completed using the well known identity

$$\binom{r-1}{j} + \binom{r-1}{j-1} = \binom{r}{j}. \quad \blacksquare$$

We are now getting ready to derive a differential equation for $p_n(x)$, the first step is

THEOREM 5.

$$p_n'(x) = A_n(x)\, p_{n-1}(x) - B_n(x)\, p_n(x),$$

where

$$A_n(x) = m \sum_{i=0}^{(m/2)-1} [\beta n^{1/m}]^{2i+1} \binom{2i}{i} x^{m-2i-2}$$

and

$$B_n(x) = m \sum_{i=0}^{(m/2)-2} [\beta n^{1/m}]^{2i+2} \binom{2i+1}{i} x^{m-2i-3} \qquad (m > 2).$$

Proof. We start with a Fourier expansion of the polynomial

$$p_n'(x) = \sum_{k=0}^{n-1} a_k\, p_k(x)$$

with

$$a_k = \int_{-\infty}^{\infty} p_n'(t)\, p_k(t) \exp(-t^m)\, dt$$

$$= m \int_{-\infty}^{\infty} p_n(t)\, p_k(t)\, t^{m-1} \exp(-t^m)\, dt.$$

$$p_n'(x) = m \int_{-\infty}^{\infty} \left(\sum_{k=0}^{n-1} p_k(x)\, p_k(t) \right) p_n(t)\, t^{m-1} \exp(-t^m)\, dt$$

$$= m \frac{\gamma_{n-1}}{\gamma_n} \int_{-\infty}^{\infty} \frac{p_n(t)\, p_{n-1}(x) - p_n(x)\, p_{n-1}(t)}{t-x}\, p_n(t)\, t^{m-1} \exp(-t^m)\, dt.$$

Since

$$\frac{t^{m-1}}{t-x} = t^{m-2} + t^{m-3}x + \cdots + x^{m-2} + \frac{x^{m-1}}{t-x}$$

and

$$\int_{-\infty}^{\infty} \frac{p_n(t)\, p_{n-1}(x) - p_n(x)\, p_{n-1}(t)}{t-x}\, p_n(t) \exp(-t^m)\, dt = 0$$

we get

$$p_n'(x) = A_n(x)\, p_{n-1}(x) - B_n(x)\, p_n(x)$$

with

$$A_n(x) = m\frac{\gamma_{n-1}}{\gamma_n}\int_{-\infty}^{\infty} p_n^2(t)\left(\sum_{j=0}^{m-2} t^{m-2-j}x^j\right)\exp(-t^m)\,dt$$

and

$$B_n(x) = m\frac{\gamma_{n-1}}{\gamma_n}\int_{-\infty}^{\infty} p_n(t)\,p_{n-1}(t)\left(\sum_{j=0}^{m-2} t^{m-2-j}x^j\right)\exp(-t^m)\,dt.$$

To complete the proof we use the evenness and oddness of $p_n(x)$ and Corollary 5 to get

$$\begin{aligned} A_n(x) &= m\frac{\gamma_{n-1}}{\gamma_n}\left(\binom{m-2}{(m-2)/2}[\beta n^{1/m}]^{m-2} \right.\\ &\quad \left. + x^2\binom{m-4}{(m-4)/2}[\beta n^{1/m}]^{m-4} + \cdots + x^{m-2}\right)\\ &= \sum_{i=0}^{(m/2)-1}\binom{2i}{i}[\beta n^{1/m}]^{2i+1}x^{m-2i-2}, \end{aligned}$$

and

$$\begin{aligned} B_n(x) &= m\frac{\gamma_{n-1}}{\gamma_n}\left(\binom{m-3}{(m-4)/2}[\beta n^{1/m}]^{m-3}x \right.\\ &\quad \left. + \binom{m-5}{(m-6)/2}[\beta n^{1/m}]^{m-5}x^3 + \cdots + \frac{\gamma_{n-1}}{\gamma_n}x^{m-3}\right)\\ &= m\sum_{i=0}^{(m/2)-2}\binom{2i+1}{i}[\beta n^{1/m}]^{2i+2}x^{m-2i-3} \qquad (m>2). \quad \blacksquare \end{aligned}$$

With Theorem 5 we may derive a differential equation for $p_n(x)$ in the form

THEOREM 6.

$$z'' + \phi(x, n)z = 0,$$

where

$$z = p_n(x)\exp\left(-\frac{x^m}{2} + h_n(x)\right)\Big/A_n^{1/2}(x)$$

and

$$\phi(x, n) = A_n^2\left(1 - \left(\frac{x}{2\beta n^{1/m}}\right)^2\right) + g_n(x).$$

The error functions h_n and g_n satisfy

$$|h_n(x)| \leqslant \frac{c_1}{n} |B_n(x)|$$

and

$$|g_n(x)| \leqslant \frac{c_2}{n} \left(B_n^2(x) + \left(1 + \frac{x^2}{n^{2/m}} \right) A_n^2(x) \right)$$

for some positive constants c_1 and c_2.

Proof. Starting with Theorem 5 we may write the following three formulas

$$p_n' = A_n p_{n-1} - B_n p_n$$
$$p_n'' = A_n' p_{n-1} + A_n p_{n-1}' - (B_n p_n)'$$

and

$$\begin{aligned} p_{n-1}' &= A_{n-1} p_{n-2} - B_{n-1} p_{n-1} \\ &= A_{n-1} \left(x \frac{\gamma_{n-1}}{\gamma_{n-2}} p_{n-1} - \frac{\gamma_{n-1}^2}{\gamma_n \gamma_{n-2}} p_n \right) - B_{n-1} p_{n-1} \\ &= \left(A_{n-1} x \frac{\gamma_{n-1}}{\gamma_{n-2}} - B_{n-1} \right) p_{n-1} - \frac{\gamma_{n-1}^2}{\gamma_n \gamma_{n-2}} A_{n-1} p_n. \end{aligned}$$

Therefore

$$\begin{aligned} p_n'' &= \left(A_n' + A_n \left(x A_{n-1} \frac{\gamma_{n-1}}{\gamma_{n-2}} - B_{n-1} \right) \right) p_{n-1} \\ &\quad - \frac{\gamma_{n-1}^2}{\gamma_n \gamma_{n-2}} A_n A_{n-1} p_n - (B_n p_n)' \\ &= \left(\frac{A_n'}{A_n} + x A_{n-1} \frac{\gamma_{n-1}}{\gamma_{n-2}} - B_{n-1} \right) (p_n' + B_n p_n) \\ &\quad - \frac{\gamma_{n-1}^2}{\gamma_n \gamma_{n-2}} A_n A_{n-1} p_n - (B_n p_n)'. \end{aligned}$$

Rewriting gives

$$\begin{aligned} p_n'' &+ \left(B_n + B_{n-1} - x A_{n-1} \frac{\gamma_{n-1}}{\gamma_{n-2}} - \frac{A_n'}{A_n} \right) p_n' + \left(\frac{\gamma_{n-1}^2}{\gamma_n \gamma_{n-2}} A_n A_{n-1} \right. \\ &\left. + B_n B_{n-1} + B_n' - x A_{n-1} B_n \frac{\gamma_{n-1}}{\gamma_{n-2}} - \frac{A_n'}{A_n} B_n \right) p_n = 0 \end{aligned}$$

or

$$p_n'' + ap_n' + bp_n = 0.$$

Letting $z = p_n \exp(\frac{1}{2}\int a\,dx)$ we arrive at

$$z'' + \phi(x, n)z = 0$$

with

$$\phi(x, n) = b - \frac{a^2}{4} - \frac{a'}{2}.$$

As new quantities have been introduced in this proof which we want to simplify we interrupt our argument at this time to make some computations.

LEMMA 2.

$$x\frac{\gamma_{n-1}}{\gamma_{n-2}} A_{n-1} - 2B_n = mx^{m-1} + h_n(x),$$

where $h_n(x)$ satisfies for some positive constant c_1

$$|h_n(x)| \leqslant c_1 n^{-1} |B_n(x)|.$$

Proof. Using the integral forms of A_{n-1} to get the first term we write

$$x\frac{\gamma_{n-1}}{\gamma_{n-2}} A_{n-1} - 2B_n = mx^{m-1} + m\sum_{i=1}^{(m/2)-1}\binom{2i}{i}[\beta n^{1/m}]^{2i}x^{m-2i-2}$$
$$- 2m\sum_{i=0}^{(m/2)-2}\binom{2i+1}{i}[\beta n^{1/m}]^{2i+2}x^{m-2i-3}.$$

From the identity $2\binom{2i+1}{i} = \binom{2i+2}{i+1}$ we get

$$x\frac{\gamma_{n-1}}{\gamma_{n-2}} A_{n-1} - 2B_n = mx^{m-1} + \sum_{i=0}^{(m/2)-2}[0][n^{1/m}]^{2i+1}x^{m-2i-3}$$

which completes the proof of Lemma 2. ∎

Another expression which we simplify is in

LEMMA 3.

$$\phi(x, n) = A_n^2\left(1 - \left(\frac{x}{2\beta n^{1/m}}\right)^2\right) + g_n(x),$$

where $g_n(x)$ satisfies for some positive constant c_2

$$|g_n(x)| \leqslant \frac{c_2}{n}\left(B_n^2(x) + \left(1 + \frac{x^2}{n^{2/m}}\right) A_n^2(x)\right).$$

Proof.

$$\begin{aligned}
\phi(x, n) &= b - \frac{a^2}{4} - \frac{a'}{2} \\
&= \frac{\gamma_{n-1}^2}{\gamma_n \gamma_{n-2}} A_n A_{n-1} + B_n B_{n-1} - x A_{n-1} B_n \frac{\gamma_{n-1}}{\gamma_{n-2}} + B_n' - \frac{A_n'}{A_n} B_n \\
&\quad - \frac{1}{4}\left(B_n + B_{n-1} - x A_{n-1} \frac{\gamma_{n-1}}{\gamma_{n-2}}\right)^2 \\
&\quad + \frac{1}{2}\left(B_n + B_{n-1} - x A_{n-1} \frac{\gamma_{n-1}}{\gamma_{n-2}}\right) \frac{A_n'}{A_n} - \frac{1}{4}\left(\frac{A_n'}{A_n}\right)^2 \\
&\quad - \frac{1}{2}\left(B_n' + B_{n-1}' - (x A_{n-1}' + A_{n-1}) \frac{\gamma_{n-1}}{\gamma_{n-2}} - \frac{A_n''}{A_n} + \left(\frac{A_n'}{A_n}\right)^2\right) \\
&= A_n^2\left(1 - \left(\frac{x}{2\beta n^{1/m}}\right)^2\right) + g_n(x).
\end{aligned}$$

In estimating $g_n(x)$ we use the following five observations that

$$\frac{|A_n''|}{A_n} \leqslant (m-1)(m-2)$$

$$\frac{|A_n'|}{A_n} \leqslant m - 1$$

$$|B_n'(x)| \leqslant \max((m-3)\,|B_n(x)|,\, B_n'(1))$$

and for some $c > 0$

$$|A_n - A_{n-1}| \leqslant \frac{c}{n} A_n$$

and

$$|B_n - B_{n-1}| \leqslant \frac{c}{n} |B_n|$$

along with many cancellations to get for some positive constant c_2

$$|g_n(x)| \leqslant \frac{c_2}{n}\left(B_n^2(x) + \left(1 + \frac{x^2}{n^{2/m}}\right) A_n^2(x)\right).$$

Lemmas 2 and 3 now complete the proof of Theorem 6. ∎

The same method used in the Hermite case can be used to obtain estimates of the Freud polynomials. A technical problem arises because there is not an exact cutoff for the change in sign of $\phi(x, n)$. Therefore in an interval around x_{1n} of size $n^{-2/3+1/m}$ we use the identity

$$z'(x) - z'(x_0) = \int_{x_0}^{x} z''(t)\, dt$$

to estimate $z'(x)$ and in turn use the same identity with z' replaced by z to estimate $z(x)$. From the error term in the differential equation of Theorem 6 we see that there is a final hurdle to overcome, namely to show that $p_n^2(x) \exp(-x^m)$ is small when x is large enough. This last detail may be completed using a method that Freud also used. First estimate $p_n(x)$ by

$$p_n^2(x) \exp(-x^m) \leqslant \gamma_n^2 x^{2n} \exp(-x^m) \qquad \text{when} \quad x > x_{1n}.$$

Then use a bound for γ_n; i.e., for some constant $\alpha > 0$

$$\gamma_n \leqslant (\alpha n!)^{-1/m}.$$

Finally, note that for any $\varepsilon > 0$ there is a $c_\varepsilon > 0$ so that

$$x^{2n} \exp(-x^m) \leqslant (\varepsilon n^{2/m})^n \qquad \text{for} \quad x > c_\varepsilon n^{1/m}.$$

We conclude with the statement of our theorem.

THEOREM 7. *Let $p_n(x)$ denote the Freud polynomial of degree n. Then, there exists positive constants C', D', and E' such that*

(i) $p_n^2(x) \exp(-x^m) \leqslant C'/\sqrt{(2\beta n^{1/m})^2 - x^2}$ *when* $|x| \leqslant 2\beta n^{1/m}$,

(ii) $\max_{x \in \mathbb{R}} p_n^2(x) \exp(-x^m) \leqslant D' n^{1/3 - 1/m}$, *and*

(iii) $\max_{x \in \mathbb{R}} p_n^2(x) \exp(-x^m) \geqslant E' n^{1/3 - 1/m}$, *for* $n = 1, 2, 3, \ldots$

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